# *Suggested Solutions to:*

# Resit Exam, Fall 2019

# Contract Theory

# February 12, 2020

This version: February 28, 2020

## Question 1: Life insurance with adverse selection

By differentiating the Lagrange function that is stated in the question, we obtain the following eight first-order condition:

$$\frac{\partial \mathcal{L}}{\partial \underline{u}_1} = 0 \Leftrightarrow \nu h'\left(\underline{u}_1\right) = \mu - \lambda, \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial \underline{u}_2^B} = 0 \Leftrightarrow \nu \underline{\theta} h'\left(\underline{u}_2^B\right) = \underline{\theta}\mu - \overline{\theta}\lambda, \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial \underline{u}_2^A} = 0 \Leftrightarrow \nu(1-\underline{\theta}) h'\left(\underline{u}_2^A\right) = (1-\underline{\theta})\mu - (1-\overline{\theta})\lambda, \tag{3}$$

$$\frac{\partial \mathcal{L}}{\partial \underline{u}_3} = 0 \Leftrightarrow \nu(1-\underline{\theta}) h'\left(\underline{u}_3\right) = (1-\underline{\theta})\mu - (1-\overline{\theta})\lambda, \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial \overline{u}_1} = 0 \Leftrightarrow (1-\nu) h'\left(\overline{u}_1\right) = \lambda, \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial \overline{u}_2^B} = 0 \Leftrightarrow (1-\nu) h'\left(\overline{u}_2^B\right) = \lambda, \tag{6}$$

$$\frac{\partial \mathcal{L}}{\partial \overline{u}_2^A} = 0 \Leftrightarrow (1-\nu) h'\left(\overline{u}_2^A\right) = \lambda, \tag{7}$$

$$\frac{\partial \mathcal{L}}{\partial \overline{u}_3} = 0 \Leftrightarrow (1-\nu) h'\left(\overline{u}_3\right) = \lambda, \tag{8}$$

By solving (1) for $\mu$ and then plugging into (2), we have

$$\nu \underline{\theta} h'\left(\underline{u}_2^B\right) = \underline{\theta}\left[\nu h'\left(\underline{u}_1\right) + \lambda\right] - \overline{\theta}\lambda \Leftrightarrow \lambda = \frac{\underline{\theta}\nu}{\overline{\theta}-\underline{\theta}}\left[h'\left(\underline{u}_1\right) - h'\left(\underline{u}_2^B\right)\right]. \tag{9}$$

Plugging this back into (1) yields

$$\mu = \nu h'\left(\underline{u}_1\right) + \frac{\underline{\theta}\nu}{\overline{\theta}-\underline{\theta}}\left[h'\left(\underline{u}_1\right) - h'\left(\underline{u}_2^B\right)\right] = \frac{\nu}{\overline{\theta}-\underline{\theta}}\left[\overline{\theta} h'\left(\underline{u}_1\right) - \underline{\theta} h'\left(\underline{u}_2^B\right)\right]. \tag{10}$$

By plugging in the above expressions for $\mu$ and $\lambda$ into (3) and then rewriting, we obtain

$$\nu(1-\underline{\theta})h'\left(\underline{u}_{23}^{A}\right) = (1-\underline{\theta})\frac{\nu}{\overline{\theta}-\underline{\theta}}\left[\overline{\theta}h'\left(\underline{u}_1\right) - \underline{\theta}h'\left(\underline{u}_2^{B}\right)\right] - (1-\overline{\theta})\frac{\underline{\theta}\nu}{\overline{\theta}-\underline{\theta}}\left[h'\left(\underline{u}_1\right) - h'\left(\underline{u}_2^{B}\right)\right]$$

or

$$(1-\underline{\theta})h'\left(\underline{u}_{23}^{A}\right) = h'\left(\underline{u}_1\right) - \underline{\theta}h'\left(\underline{u}_2^{B}\right). \tag{11}$$

Similarly, the right-hand sides of (5)-(8) are identical, which means that $\overline{u}_1 = \overline{u}_2^B = \overline{u}_2^A = \overline{u}_3 \equiv \overline{u}$. By plugging in the above expression for $\lambda$ in (5), we obtain

$$(1-\nu)h'\left(\overline{u}\right) = \frac{\underline{\theta}\nu}{\overline{\theta}-\underline{\theta}}\left[h'\left(\underline{u}_1\right) - h'\left(\underline{u}_2^{B}\right)\right] \Leftrightarrow \left(\overline{\theta}-\underline{\theta}\right)(1-\nu)h'\left(\overline{u}\right) = \underline{\theta}\nu\left[h'\left(\underline{u}_1\right) - h'\left(\underline{u}_2^{B}\right)\right]. \tag{12}$$

We also know that IR-low and IC-high bind. The binding IR-low constraint can be written as

$$\underline{u}_1 + \underline{\theta}\underline{u}_2^B + 2(1-\underline{\theta})\underline{u}_{23}^A = \underline{U}^*. \tag{13}$$

**Part (a)**

The right-hand sides of (5)-(8) are identical. Moreover, the function $h'$ is strictly increasing. Therefore, the arguments of $h'$ on the left-hand sides of (5)-(8) must all equal each other. That is, at the optimum we must have

$$\overline{u}_1 = \overline{u}_2^B = \overline{u}_2^A = \overline{u}_3 \stackrel{\text{def}}{=} \overline{u}. \tag{14}$$

which means that the high type is fully insured at the optimum.

**Part (b)**

First note that the right-hand sides of (3) and (4) are identical, which means that $\underline{u}_2^A = \underline{u}_3 \stackrel{\text{def}}{=} \underline{u}_{23}^A$. Next, by solving (1) for $\mu$ and then plugging into (2), we have

$$\nu\underline{\theta}h'\left(\underline{u}_2^{B}\right) = \underline{\theta}\left[\nu h'\left(\underline{u}_1\right) + \lambda\right] - \overline{\theta}\lambda \Leftrightarrow \lambda = \frac{\underline{\theta}\nu}{\overline{\theta}-\underline{\theta}}\left[h'\left(\underline{u}_1\right) - h'\left(\underline{u}_2^{B}\right)\right]. \tag{15}$$

We know from any one of equations (5)-(8) that $\lambda > 0$. Therefore, (15) implies $h'\left(\underline{u}_1\right) > h'\left(\underline{u}_2^B\right)$, which in turn (since $h'' > 0$) means that $\underline{u}_1 > \underline{u}_2^B$. Finally, by dividing both sides of (3) by $1-\underline{\theta}$, we have

$$\nu h'\left(\underline{u}_2^{A}\right) = \mu - \frac{1-\overline{\theta}}{1-\underline{\theta}}\lambda, \tag{16}$$

the right-hand side of which is larger than $\mu - \lambda$. Combining (16) and (1) therefore yields $h'\left(\underline{u}_2^A\right) > h'\left(\underline{u}_1\right)$, which in turn (thanks to $h'' > 0$) means that $\underline{u}_2^A > \underline{u}_1$ . All in all, we have shown that, at the optimum,

$$\underline{u}_2^A = \underline{u}_3 > \underline{u}_1 > \underline{u}_2^B. \tag{17}$$

**Part (c)**

Standard arguments. See lecture slides and textbook.

## Question 2: Moral hazard with mean-variance preferences

**(a)** Solve for the $\beta$ parameter in the second-best optimal contract, denoted by $\beta^{SB}$ (you do not need to solve for $\alpha^{SB}$, and you will not get any credit if you nevertheless do that). You should make use of the following (well-known) result:

$$EU = -\exp\left[-r\left(\alpha + \beta e - \frac{1}{2}e^2 - \frac{1}{2}\nu r \beta^2\right)\right].$$

- P chooses the parameters in the contract, $\alpha$ and $\beta$. In addition, P can effectively choose A's effort $e$, because P designs the incentives that A faces when deciding what effort to make. We can thus think of P as choosing $\alpha$, $\beta$, and $e$ in order to maximize his expected payoff, subject to A's individual rationality (IR) constraint and incentive compatibility (IC) constraint. P's problem:

$$\max_{\alpha,\beta,e}\left\{\overbrace{(1-\beta)\,e-\alpha}^{=EV}\right\}$$

subject to

$$\overbrace{-\int_{-\infty}^{\infty}\exp\left[-r\left(t-c\left(e\right)\right)\right]f\left(z\right)dz}^{=EU} \geq -\exp\left[-r\hat{t}\right], \tag{IR}$$

$$e \in \arg\max_{e'} EU\left(e'\right). \tag{IC}$$

The IC constraint says that $e$ indeed maximizes A's utility among all the $e$'s that A could choose. The IR constraint says that A's expected utility if accepting the contract is at least as large as his utility from his outside option; this therefore ensures that A wants to participate.

- The IC constraint above is actually a whole set of infinitely many constraints. In order to reduce these to one single IC constraint, we can make use of the first-order approach, which means that we replace IC above with the first-order condition from A's maximization problem (for some arbitrary values of the contract parameters $\alpha$ and $\beta$). From the question we have that A's expected utility can be written as

$$EU = -\exp\left[-r\left(\alpha + \beta e - \frac{1}{2}e^2 - \frac{1}{2}\nu r \beta^2\right)\right].$$

Maximizing $EU$ is equivalent to maximizing a monotone transformation of this expression, so we can without loss of generality let A maximize

$$\widetilde{EU} = \alpha + \beta e - \frac{1}{2}e^2 - \frac{1}{2}\nu r \beta^2. \tag{18}$$

- We have

$$\frac{\partial \widetilde{EU}}{\partial e} = \beta - e = 0$$

Therefore A's optimal effort level is

$$e = \beta. \tag{19}$$

- We can write the IR constraint as

$$
\begin{aligned}
-\int_{-\infty}^{\infty} \exp\left[-r\left(t-c\left(e\right)\right)\right] f\left(z\right) dz \geq -\exp\left[-r\widehat{t}\right] &\Leftrightarrow \\
-\exp\left[-r\left(\alpha+\beta e-\frac{1}{2}e^2-\frac{1}{2}\nu r\beta^2\right)\right] \geq -\exp\left[-r\widehat{t}\right] &\Leftrightarrow \\
\exp\left[-r\left(\alpha+\beta e-\frac{1}{2}e^2-\frac{1}{2}\nu r\beta^2\right)\right] \leq \exp\left[-r\widehat{t}\right] &\Leftrightarrow \\
-r\left(\alpha+\beta e-\frac{1}{2}e^2-\frac{1}{2}\nu r\beta^2\right) \leq -r\widehat{t} &\Leftrightarrow \\
\alpha+\beta e-\frac{1}{2}e^2-\frac{1}{2}\nu r\beta^2 \geq \widehat{t} &\Leftrightarrow \\
\alpha \geq \widehat{t}-\beta e+\frac{1}{2}e^2+\frac{1}{2}\nu r\beta^2&.
\end{aligned}
$$

Plugging in (19) in this inequality, we obtain

$$
\begin{aligned}
\alpha &\geq \widehat{t}-\beta^2+\frac{1}{2}\beta^2+\frac{1}{2}\nu r\beta^2 \\
&= \widehat{t}-\frac{1}{2}\left(1-\nu r\right)\beta^2.
\end{aligned}
$$

Plugging in (19) into P's objective function $EV = \left(1-\beta\right)e-\alpha$ , we have

$$
EV = \left(1-\beta\right)\beta-\alpha.
$$

- Using the above results, P's problem becomes

$$
\max_{\alpha,\beta}\left\{\left(1-\beta\right)\beta-\alpha\right\} \qquad \text{subject to}
$$

$$
\alpha \geq \widehat{t}-\frac{1}{2}\left(1-\nu r\right)\beta^2. \tag{IR}
$$

- It is clear that IR must bind, as the objective is decreasing in $\alpha$ and the constraint is tightened as $\alpha$ is lowered (thus P wants to lower $\alpha$ until the constraint says stop). We thus have $\alpha = \widehat{t}-\frac{1}{2}\left(1-\nu r\right)\beta^2$. Plugging this value of $\alpha$ into the objective yields the following unconstrained problem:

$$
\max_{\beta}\left\{\beta-\frac{1}{2}\left(1+\nu r\right)\beta^2-\widehat{t}\right\},
$$

with the first-order condition

$$
1-\left(1+\nu r\right)\beta = 0 \Rightarrow \beta^{SB} = \frac{1}{1+\nu r}.
$$

**(b)** Does the agent get any rents at the second-best optimum? Do not only answer yes or no, but also explain how you can tell.

- No, he does not get any rents at the second-best optimum. "Rents" are defined as any payoff from accepting the contract that exceeds the outside option payoff. However, we saw under a) that the IR constraint binds at the optimum, which means that A does not get any rents.

**(c)** The first-best values of the effort level and the $\beta$ parameter equal $e^{FB} = 1$ and $\beta^{FB} = 0$, respectively. How do these values relate to the corresponding second-best values? In particular, is there under- or overprovision of effort at the second-best optimum?

- We have from the above analysis that $\beta^{SB} = e^{SB} = \frac{1}{1+\nu r}$. We see that there is underprovision of effort (as $e^{SB} < e^{FB}$). We also see that the $\beta$ parameter is too large relative to the first best level ($\beta^{SB} > \beta^{FB}$).

**(d)** Consider the limit case where $r \to 0$. Explain what happens to the relationship between the second-best and the first-best effort levels. Also explain the intuition for this result.

- In the limit where $r \to 0$, A is risk neutral. We see from above that in that limit, $e^{SB} = 1$. That is, the second-best effort level coincides with the first-best level: there is no inefficiency in spite of the fact that there is asymmetric information. The reason why this can occur is that when A is risk neutral he doesn't mind bearing risk. Therefore P can incentivize A very strongly, so that indeed $\beta^{SB} \to 1$ as $r \to 0$: A's compensation depends fully on the stochastic variable, so he makes the same decision as P would have made if he had been in A's job.
- The intuition is the same as we have discussed in other parts of the course, for example in the 2x2 moral hazard model with a risk neutral agent who is not protected by limited liability. There we explained the intuition as follows:
  - The economic meaning of the fact that A is risk neutral is that he cares only about whether his payment $t$ is large enough *on average*. Hence, P can, without violating the participation constraint, incentivize A by giving him a negative payment (in practice a penalty) in case of a low output. More generally, P can achieve the first-best outcome by making A the residual claimant:
    * Then A effectively buys the right to receive any returns: "the firm is sold to the agent".
    * Thereby, the effort level is chosen by the same individual who bears the consequences of the choice.
    * In this situation A makes the same effort choice as P would have made.